CROSSED PRODUCTS BY FINITE GROUPS: LECTURES AT THE 2016 SHANGHAI SUMMER SCHOOL

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The purpose of this lecture series is to give an introduction to the theory of crossed product C*-algebras. It is aimed at people familiar with the material of a first course on operator algebras, including most of:

- The characterization of commutative C*-algebras as algebras of continuous functions.
- The characterization of general C*-algebras as algebras of operators on Hilbert space.
- The structure of finite dimensional C*-algebras.
- Continuous functional calculus.
- Approximate identities.
- Ideals and quotients.
- Murray-von Neumann equivalence of projections.
- The basics of representation theory on Hilbert spaces.
- Hereditary subalgebras.
- Direct limits.
- Familiarity with some examples. (UHF algebras, AF algebras, and irrational rotation algebras are particularly helpful.)
- Stable rank and real rank (mostly the special cases stable rank one and real rank zero).

It isn't essential to have all of these; a few missing things can be filled in as we go, or taken on faith. Continuous functional calculus is essential. We will use tracial rank zero; the definition will be given and discussed in the lectures. Knowledge of semiprojectivity and K-theory is helpful for motivation, but is not necessary.

The basic theory of general crossed products is somewhat technical. We will stick to crossed products by finite groups. When the group is discrete (and the algebra is unital), the crossed product contains unitaries corresponding to the group elements; in particular, it is unital. Requiring that the group be finite also avoids the need for completion: one can write down all elements of the crossed product explicitly.

The main results to be proved in the lectures are the basic theory of crossed products of unital C*-algebras by finite groups, followed by the following two theorems.

Theorem 1. Let A be a unital AF algebra. Let G be a finite group, and let $\alpha\colon G \to \mathrm{Aut}(A)$ have the Rokhlin property. Then $C^*(G, A, \alpha)$ is an AF algebra.

Theorem 2. Let A be a simple separable unital C*-algebra with tracial rank zero. Let G be a finite group, and let $\alpha\colon G \to \mathrm{Aut}(A)$ have the tracial Rokhlin property. Then $C^*(G, A, \alpha)$ has tracial rank zero.

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Some applications will be stated, such as to crossed products of irrational rotation algebras by actions of finite subgroups of $\mathrm{SL}_2(\mathbb{Z})$.

Numerous exercises, of varying levels of difficulty, will be given in the lectures. I strongly recommend doing most of them, to solidify your understanding of the material.

Here is a rough outline of all five lectures:

- Actions of finite groups on C*-algebras and examples.
- Crossed products by actions of finite groups: elementary theory.
- Crossed products by actions of finite groups: some examples.
- The Rokhlin property for actions of finite groups.
- Examples of actions with the Rokhlin property.
- Crossed products of AF algebras by actions with the Rokhlin property.
- Other crossed products by actions with the Rokhlin property.
- The tracial Rokhlin property for actions of finite groups.
- Examples of actions with the tracial Rokhlin property.
- Crossed products by actions with the tracial Rokhlin property.
- Applications of the tracial Rokhlin property.
- Other results and open problems.

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